A Greedy Algorithm for Low-Crossing Partitions for General Set Systems

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Preliminaries

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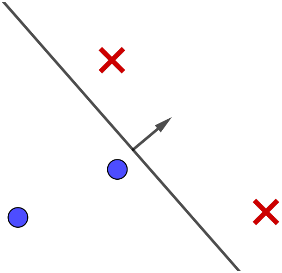

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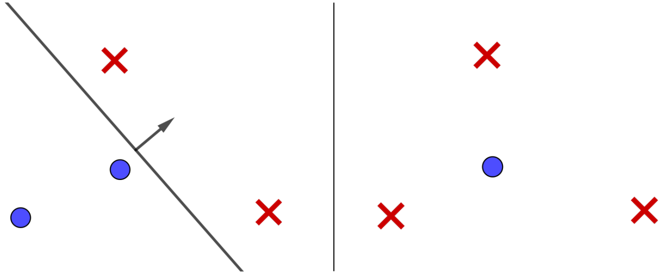

Low-crossing partitions

Low-crossing Partitions

Given $(X, \mathcal{F})$ and a parameter $t \in [2..\frac{n}{2}]$, find a partition of X in t parts $P_1, .., P_t$ of size $O\left(\frac{n}{t}\right)$ to minimize the *crossing number*:

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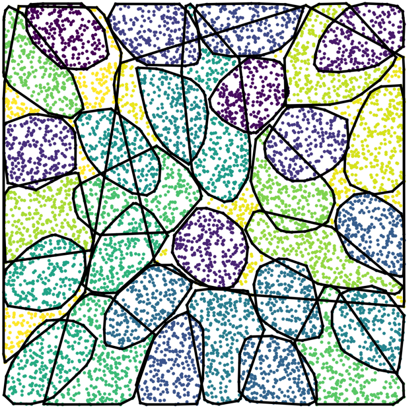

A partition of points in the unit cube and random halfspaces

Previous work

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There are neither fast algorithms in practice in dimension > 3 to compute small crossing number partitions for geometric set systems nor for abstract set systems in any dimension.

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Then there exists an ordering of the elements of P_l, say $\langle x_1, x_2, \ldots, x_{n/t} \rangle$, such that w.h.p.,

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An algorithm that picks elements following this potential function builds partitions with crossing number $O(t^{1-1/d} \ln(t))$.

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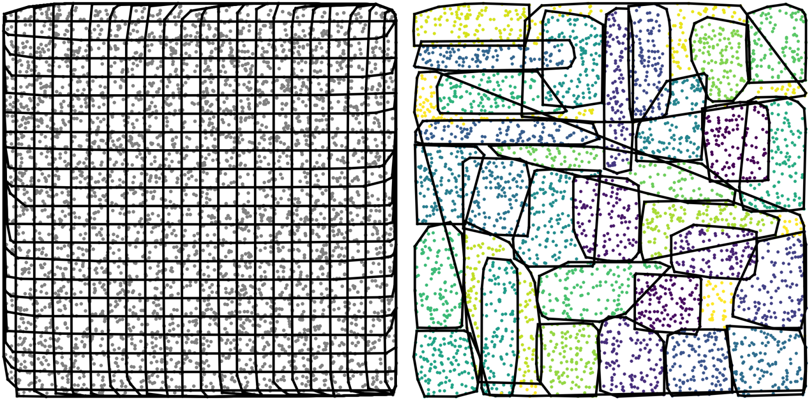
- We double the weight of ranges crossing the part built.

Experiments

Experimental Setup

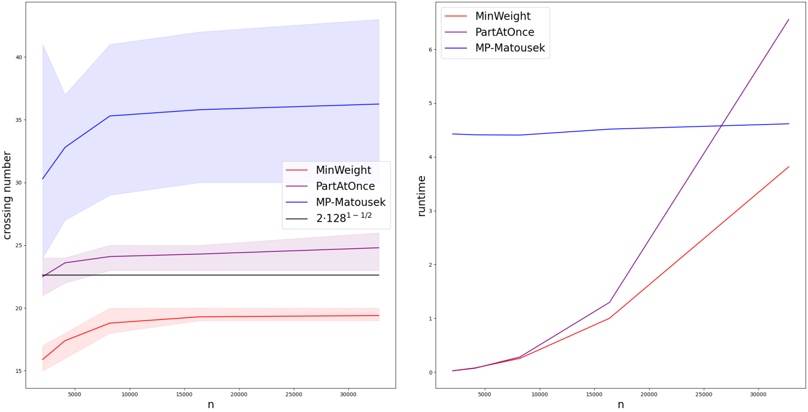
- We propose two variations of the greedy algorithms:
  - → MinWeight: We pick the element minimizing cost at each iteration.
  - → PartAtOnce: We estimate weights with range sampling and pick the n/t elements with lowest estimated weight as a part.
- Experiments results are obtained from average over 10 runs.
- Experiments have been performed on AMD Ryzen 7 5800X (16 cores) @ 4.85 GHz (Home computer).

Experiments



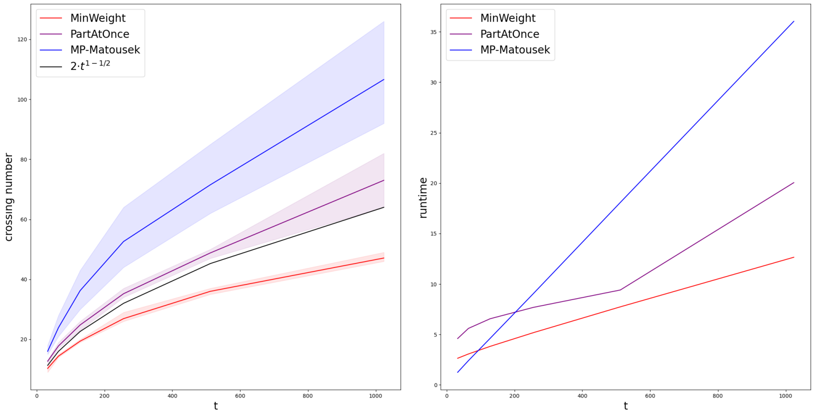
$n = 8192$, $t = 32$, crossing number : 10

Comparison with previous methods (n)



Comparison of crossing number and runtime of our algorithms for varying n and $t = 128, d = 2$.

Comparison with previous methods (t)



Comparison of crossing number and runtime of our algorithms for varying t and $n = 32768, d = 2$.

Real world datasets: ArXiv co-authorship graph

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Input t	$t^{4/5}$	MinWeight $\kappa_{\mathcal{F}}$	MinWeight runtime (s)	PartAtOnce $\kappa_{\mathcal{F}}$	PartAtOnce runtime (s)
50	23	12	0.973	21	0.182
100	40	16	1.08	28	0.256
200	69	19	1.17	34	0.373
500	144	26	1.46	41	0.74

VC-dimension: 5 [CCDV24]

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The code is available online
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Thank You!

More results: ε-approximations

ε − *approximation*

Given $(X, \mathcal{F})$, an ε-approximation is a set A such that:

$$\forall F \in \mathcal{F}, \left| \frac{|F|}{|X|} - \frac{|F \cap A|}{|A|} \right| \leq \varepsilon$$

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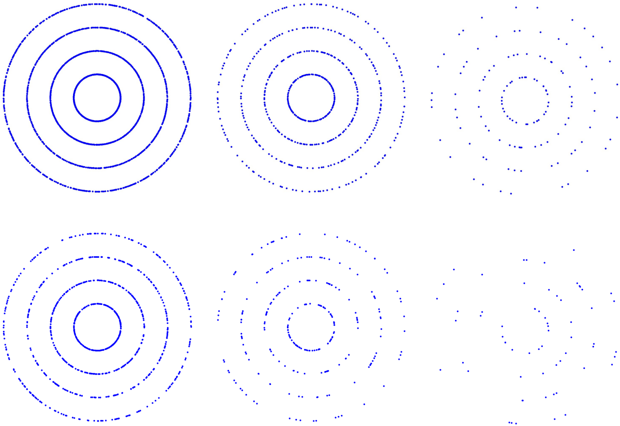
[STZ06] proved that it is possible to construct an ε-approximation of size $O\left(\frac{d}{\varepsilon^{\frac{2d}{d+1}}}\right)$ using low-crossing partitions for set systems with VC-dim $\leq d$.

ε-approximation experiments



our algorithm

random sample [Tal94]

$|A| = \frac{n}{4}$ $|A| = \frac{n}{16}$ $|A| = \frac{n}{64}$

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