An example of hypergraph approximation and some prospective differential privacy questions

Alexandre Louvet

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INRIA Lille

Set systems/Hypergraphs

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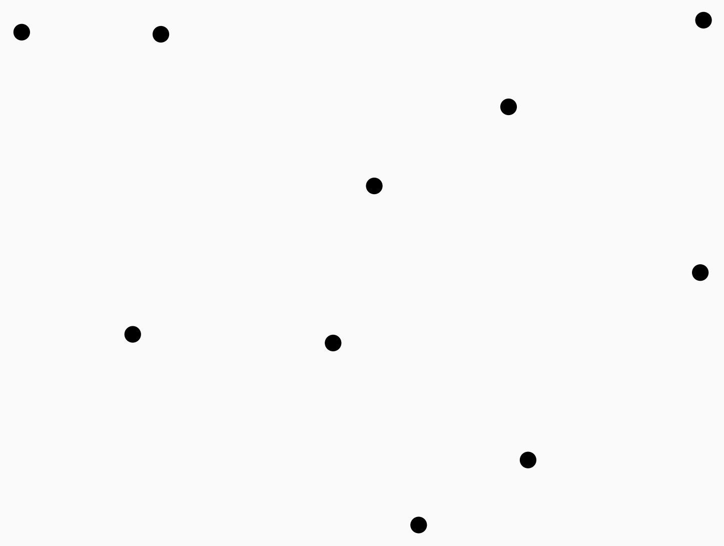

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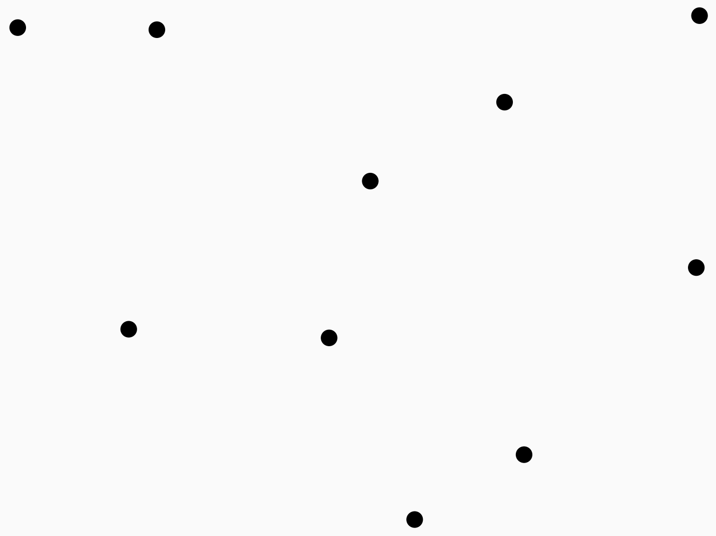

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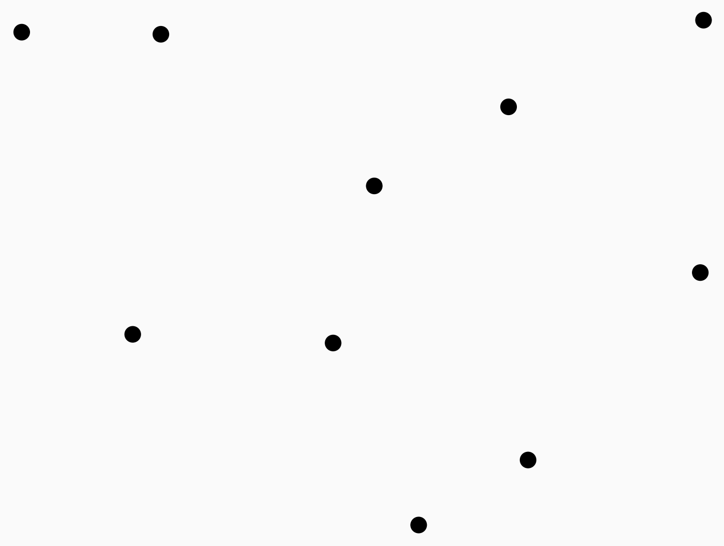

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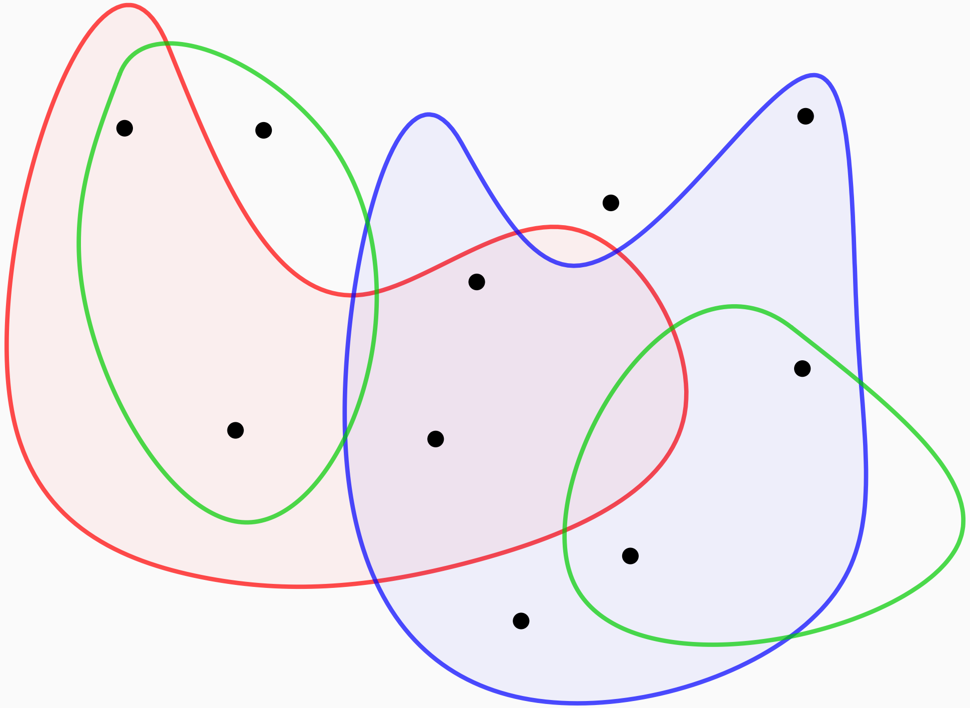

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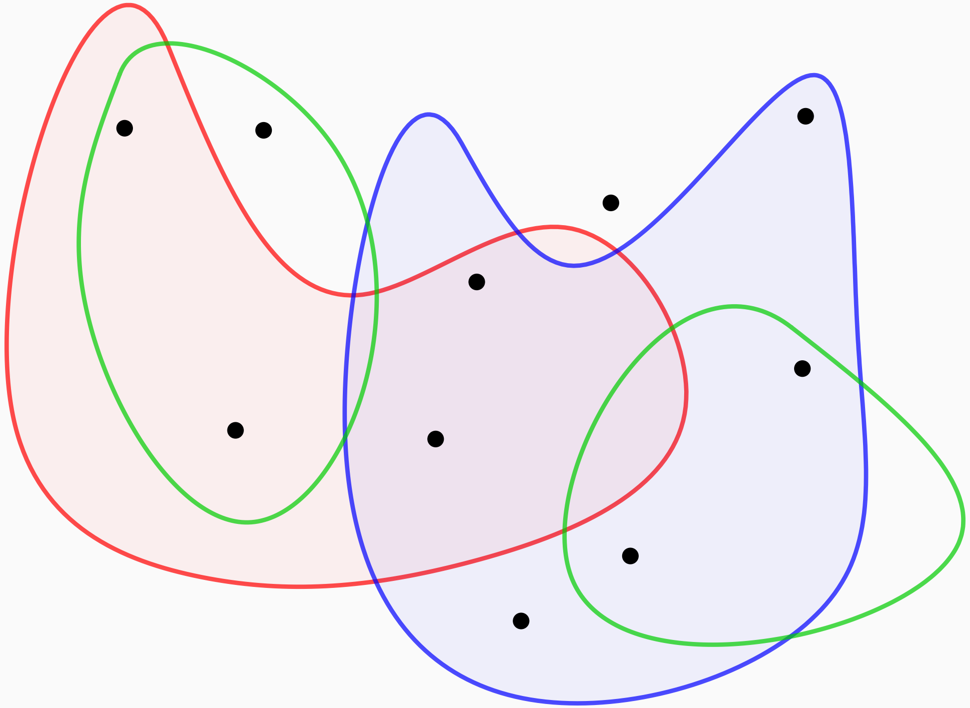

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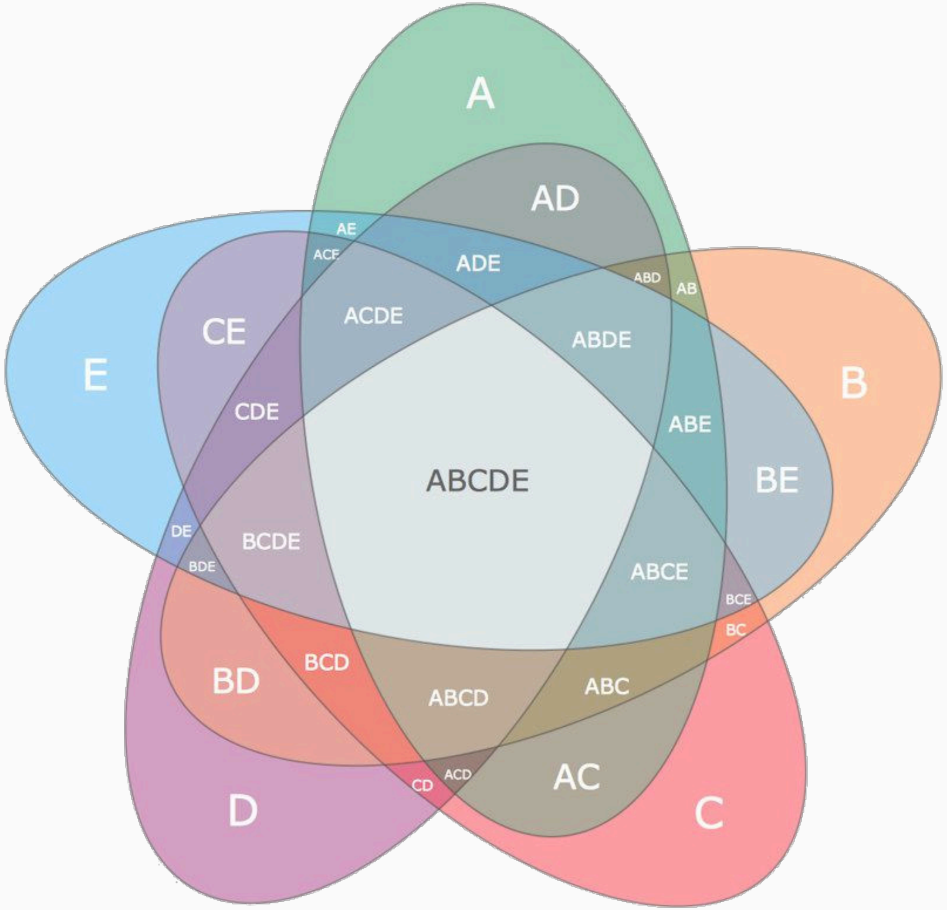

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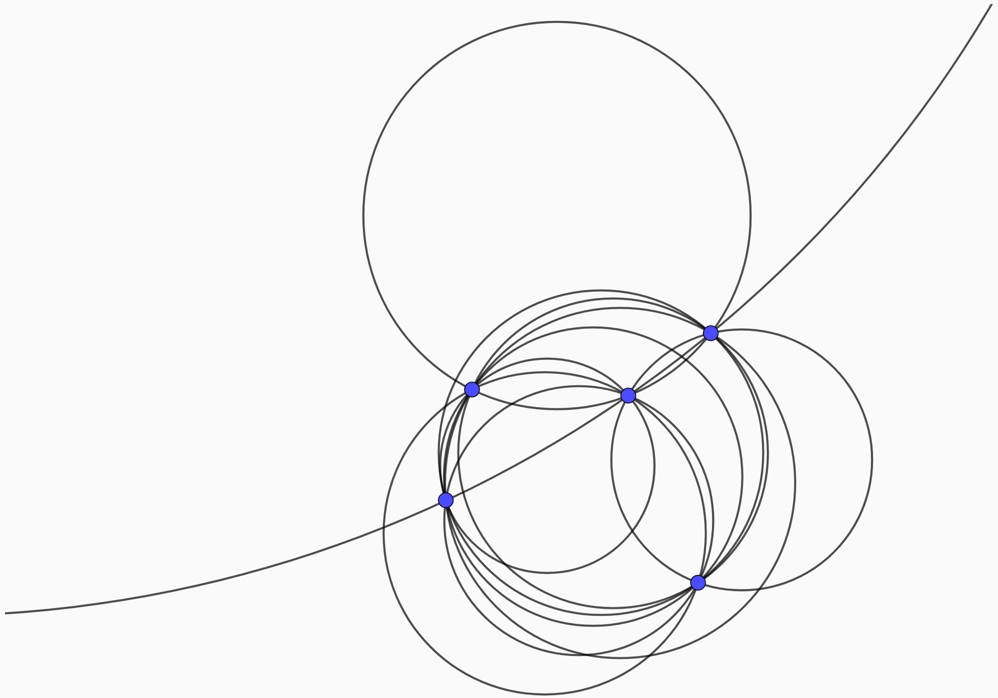

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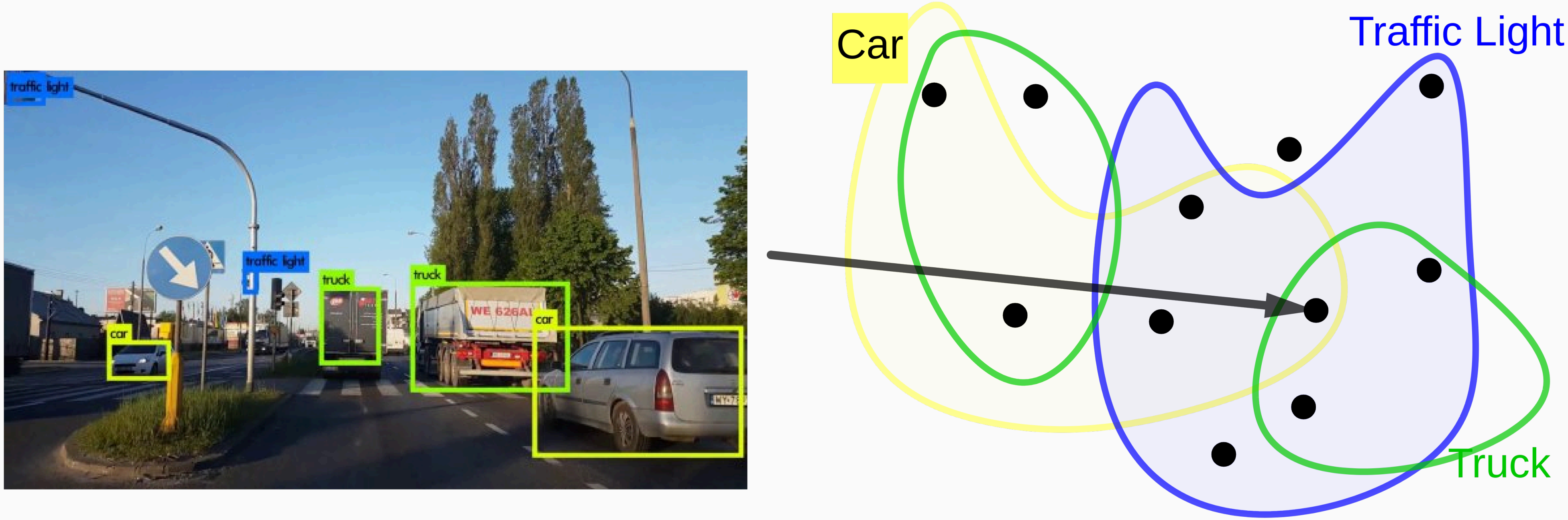

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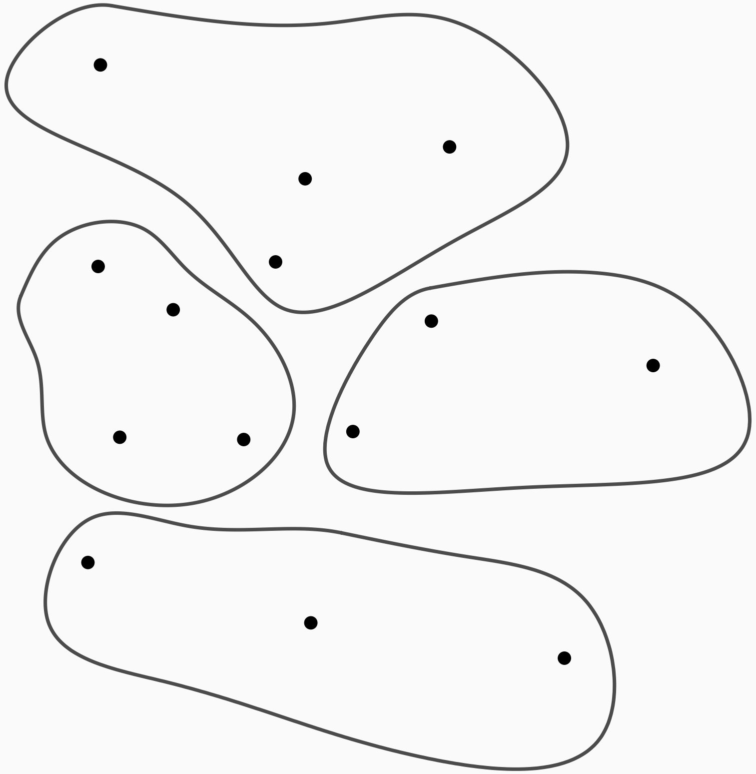

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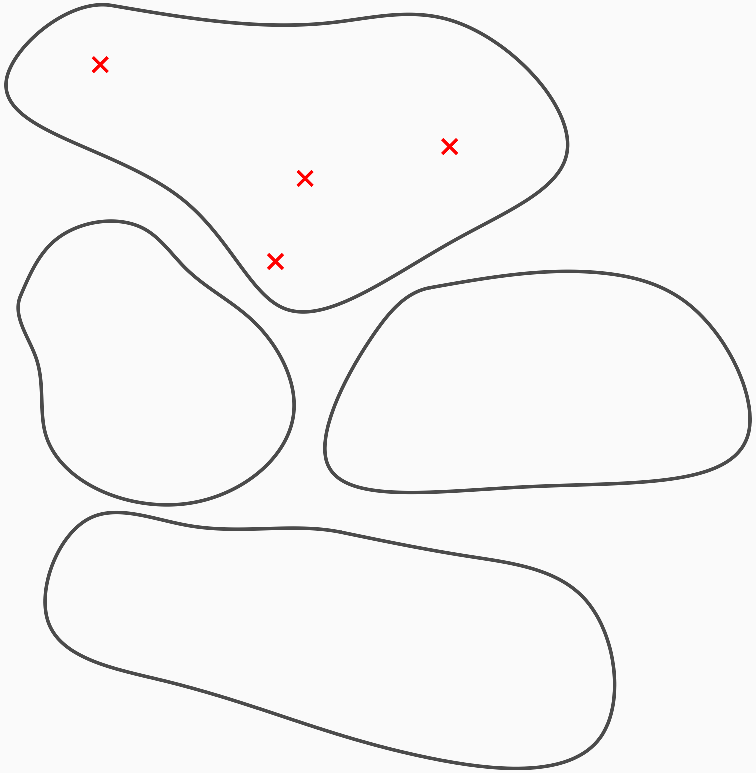

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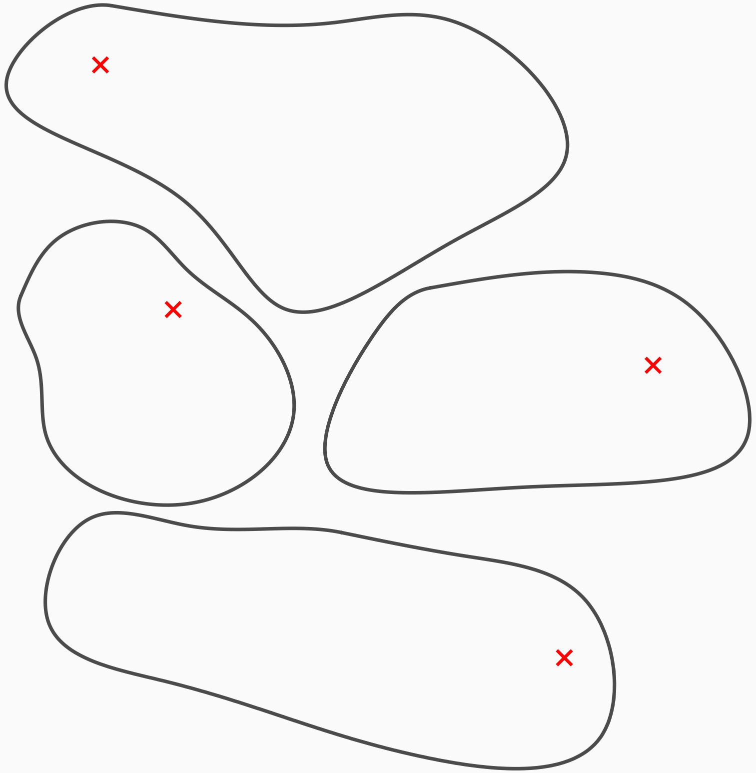

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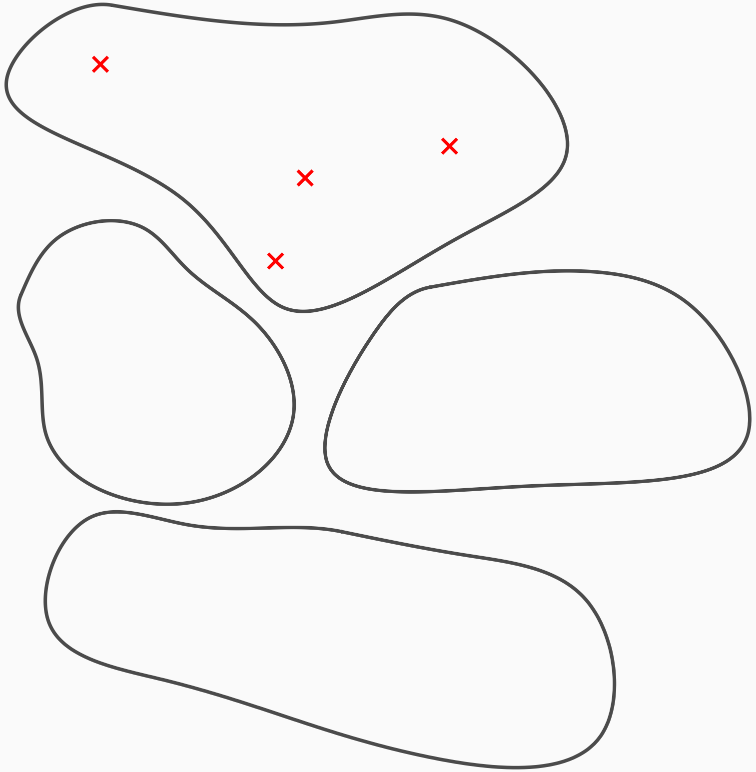

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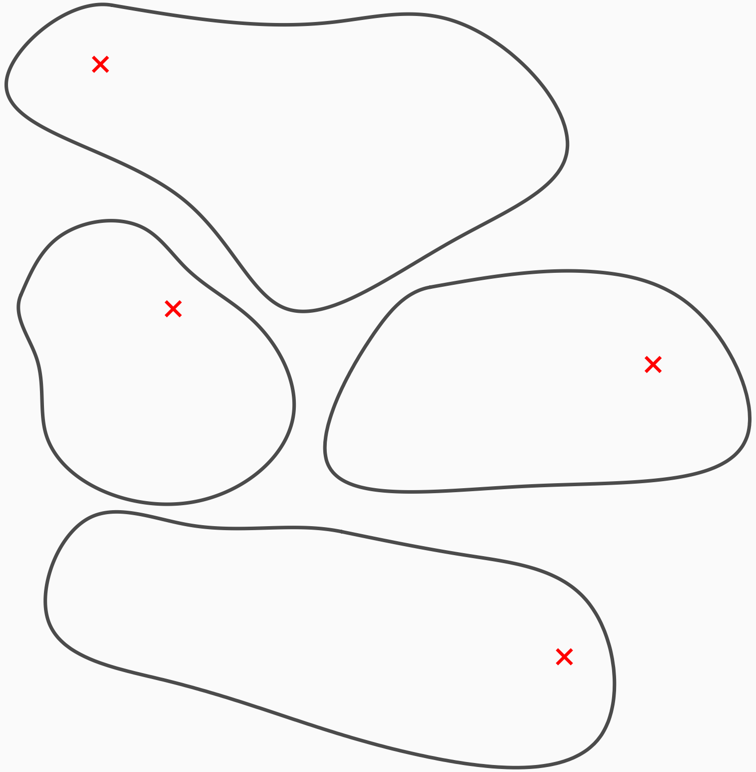

$$\varepsilon = \frac{1}{21}$$

Combinatorial discrepancy

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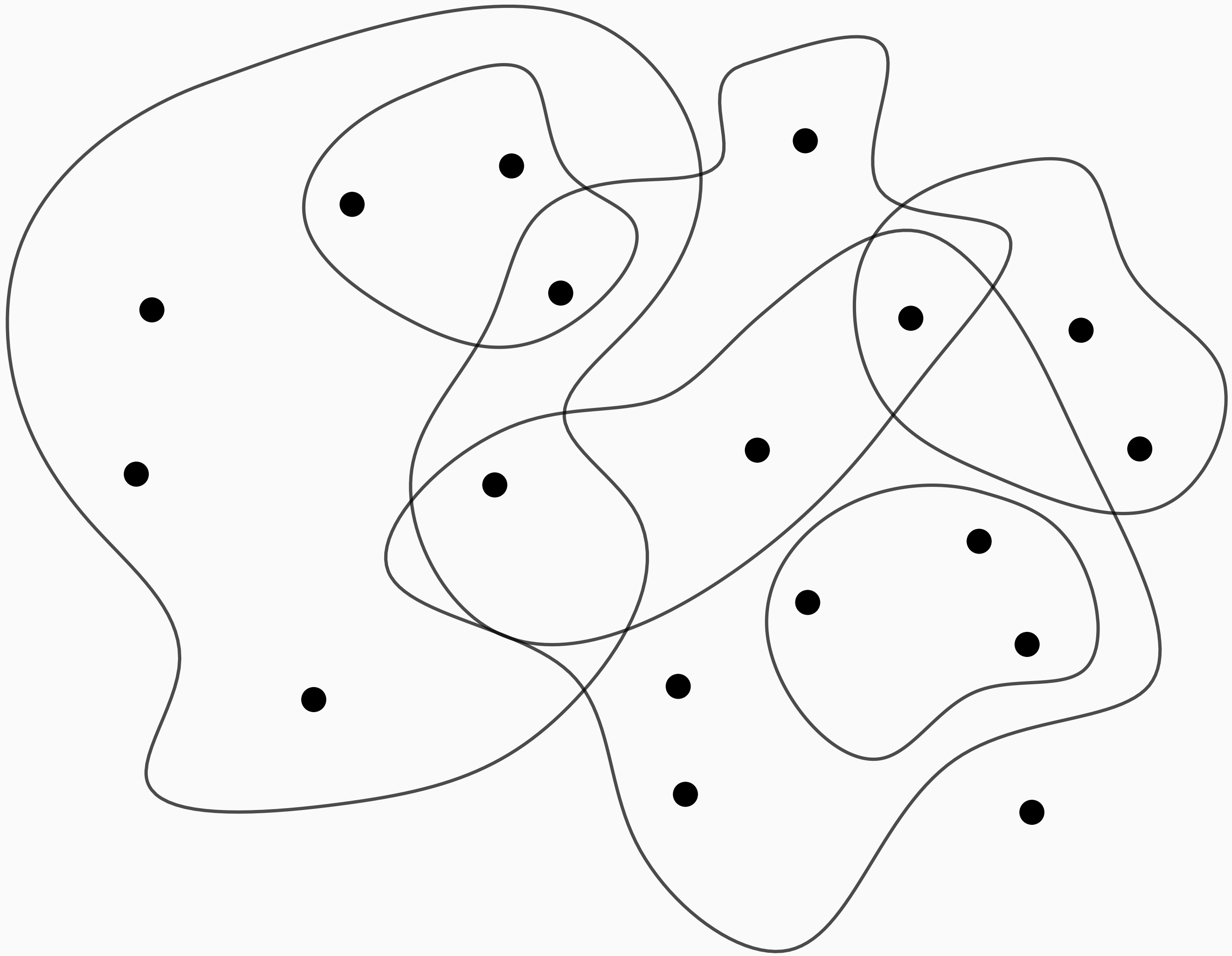

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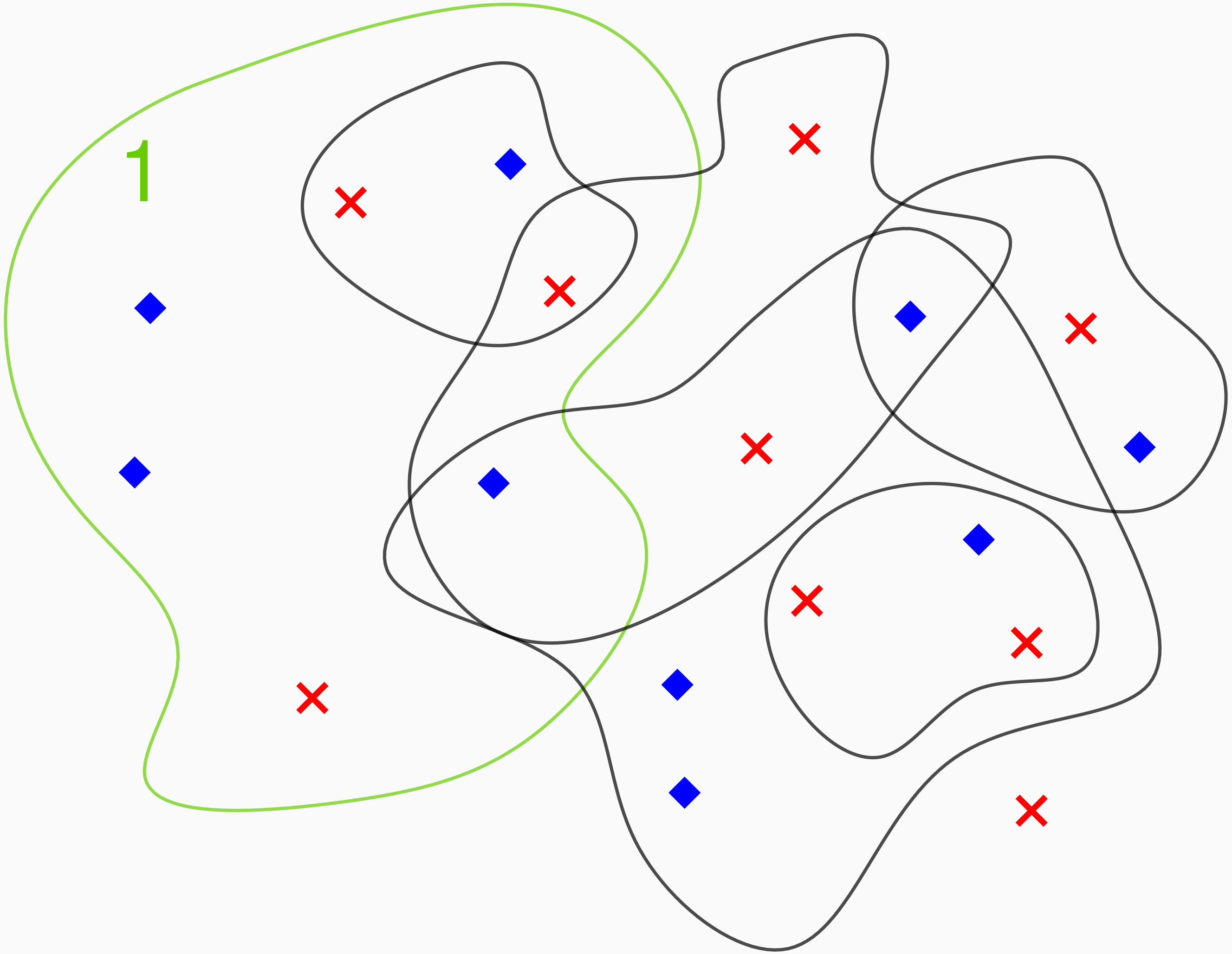

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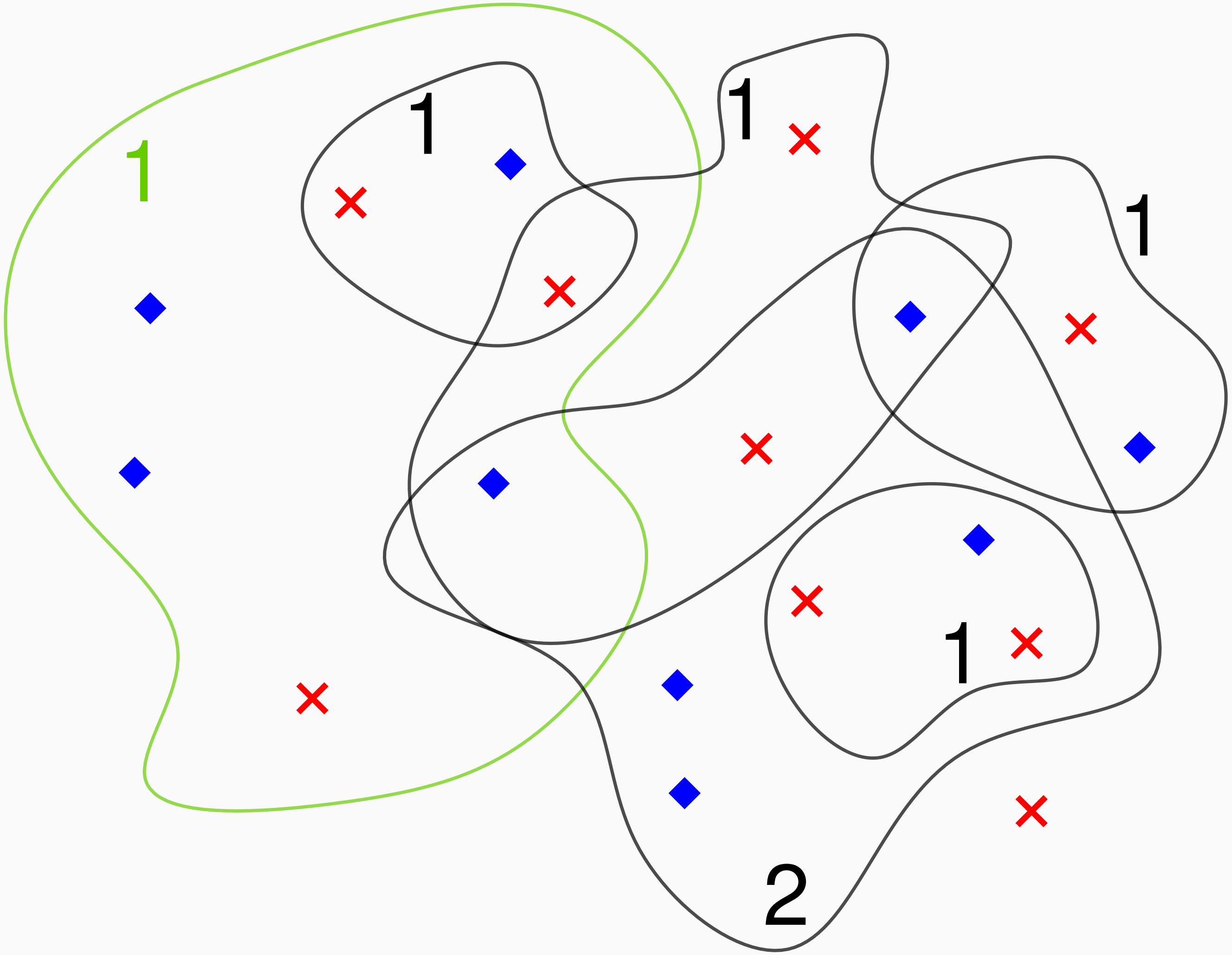

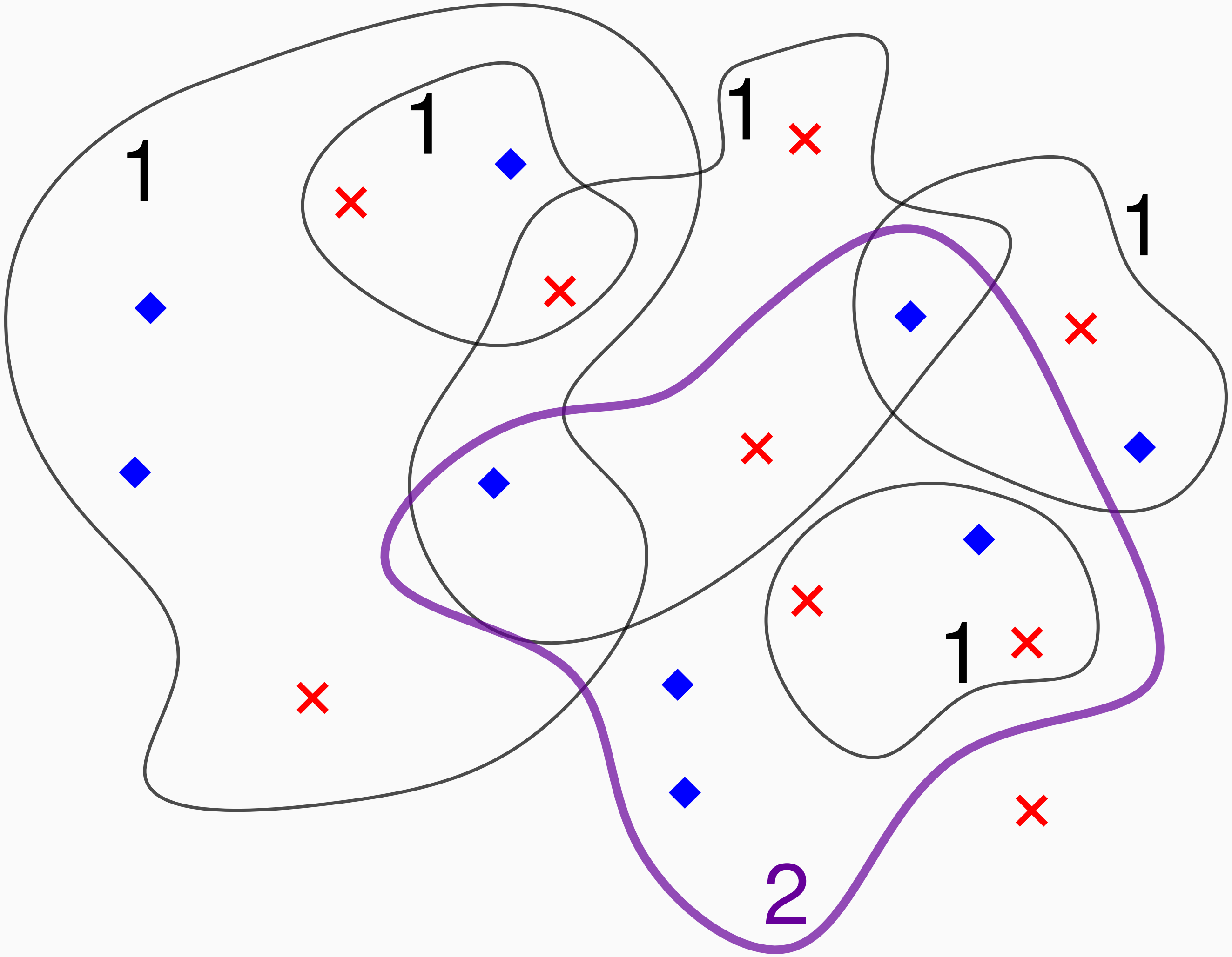
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Goal: Compute a small discrepancy coloring

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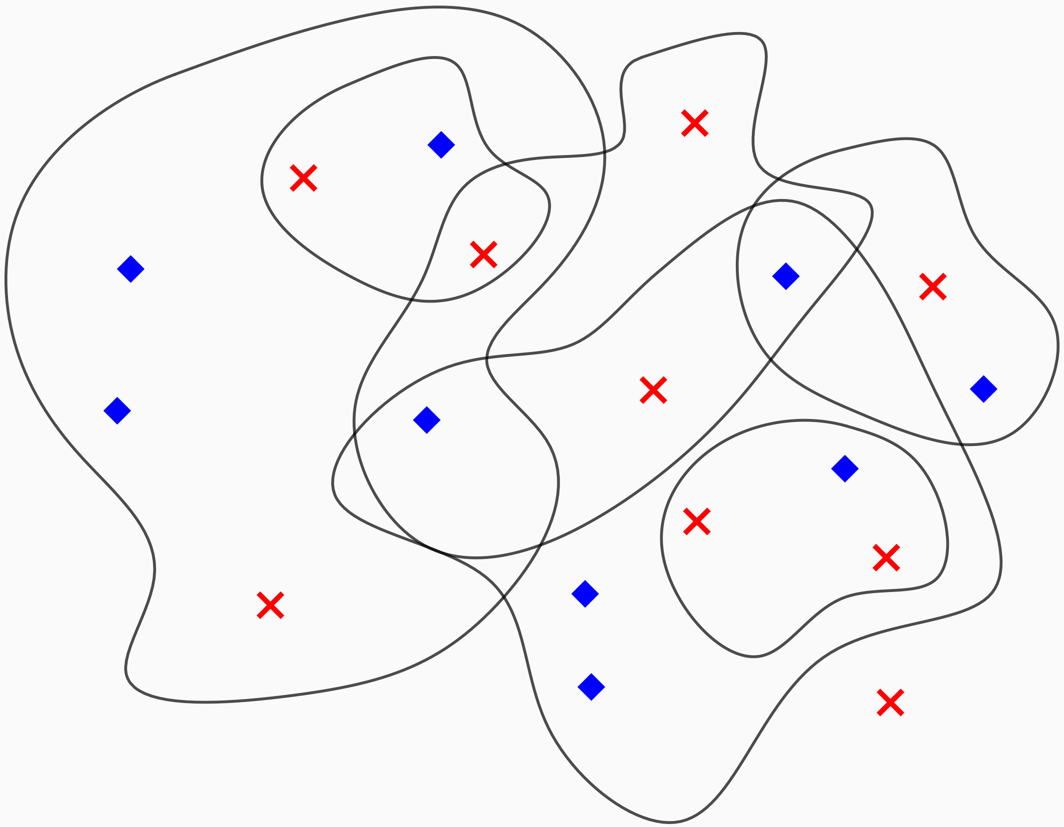

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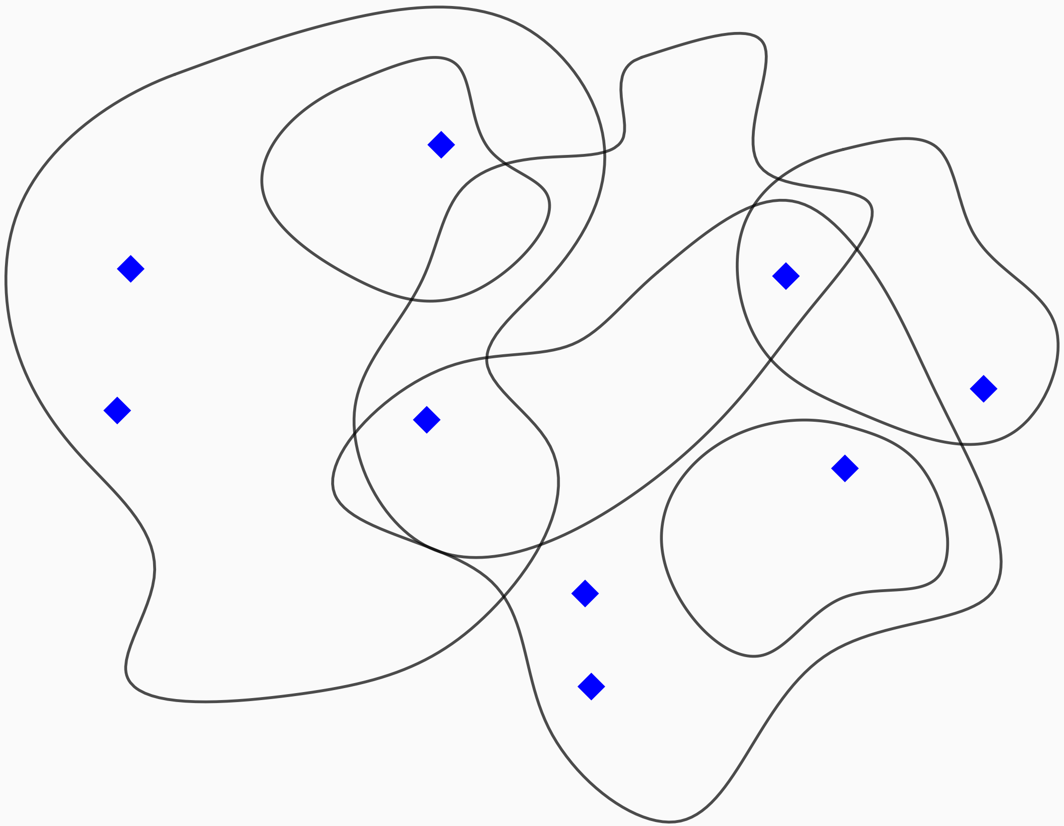

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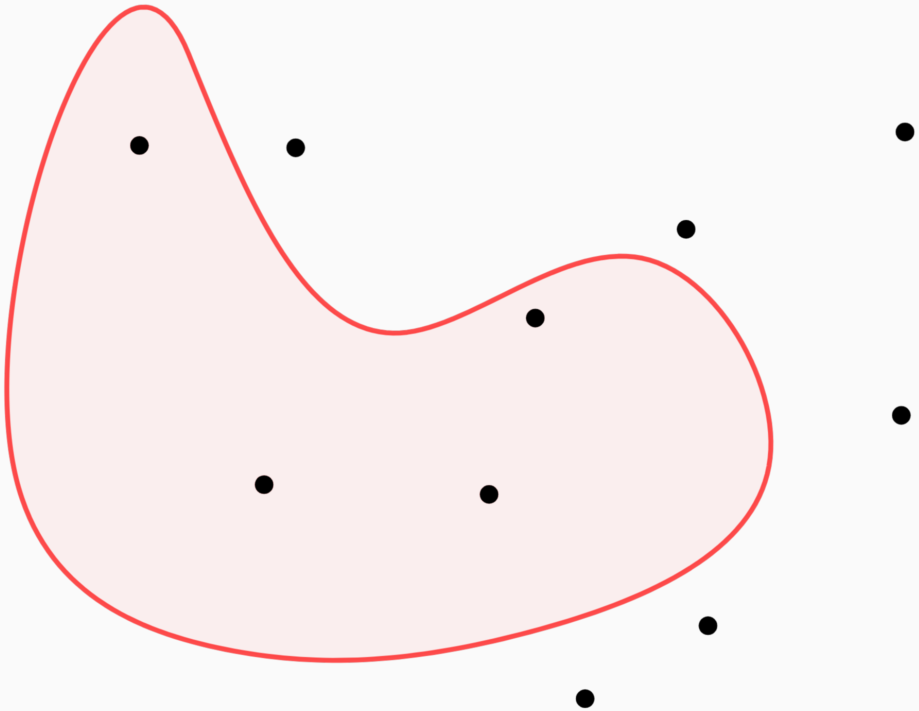

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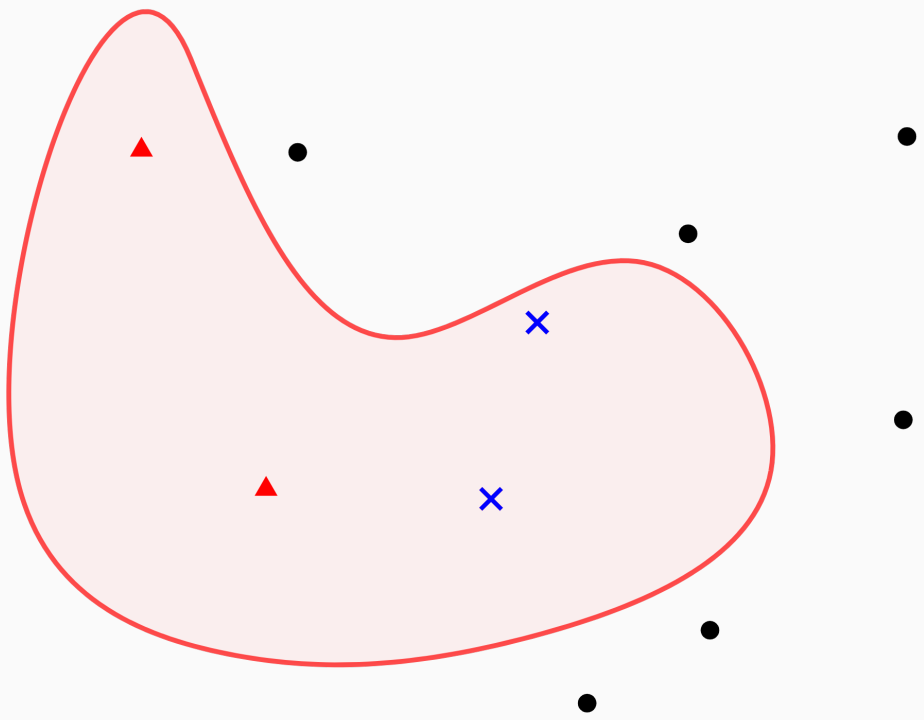

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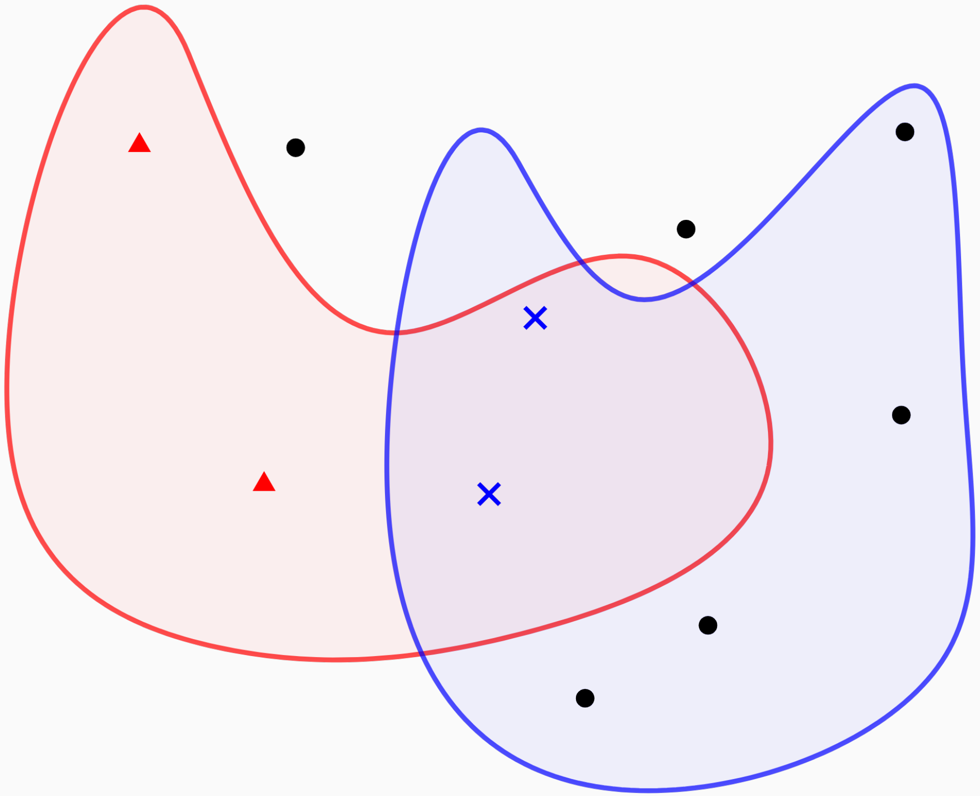

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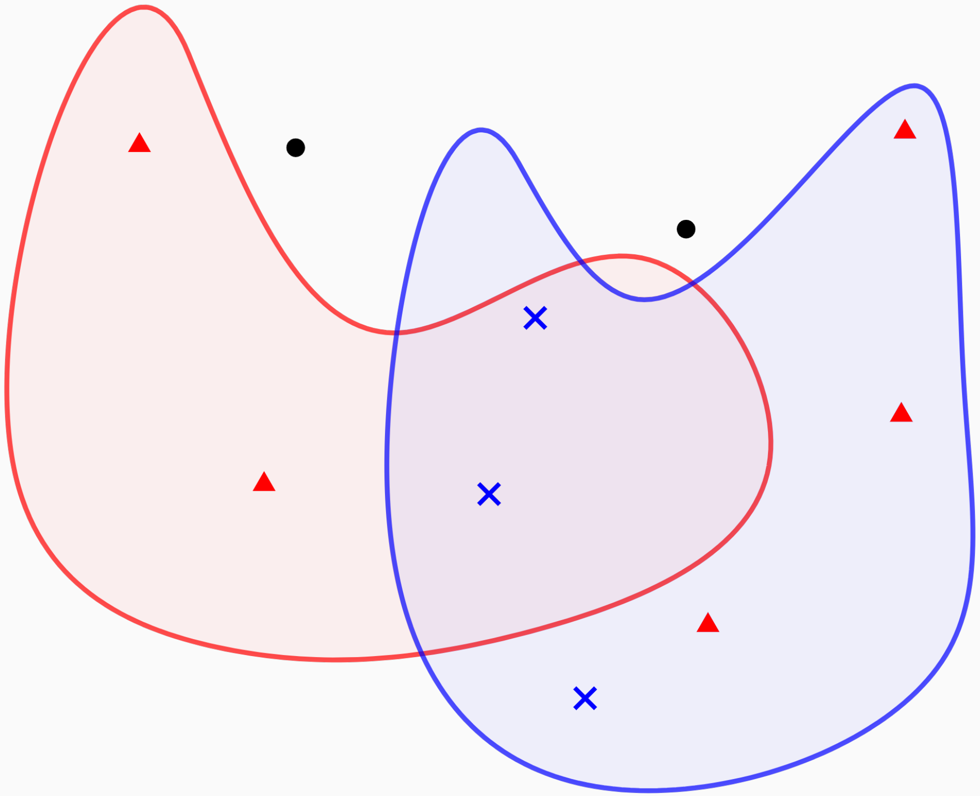

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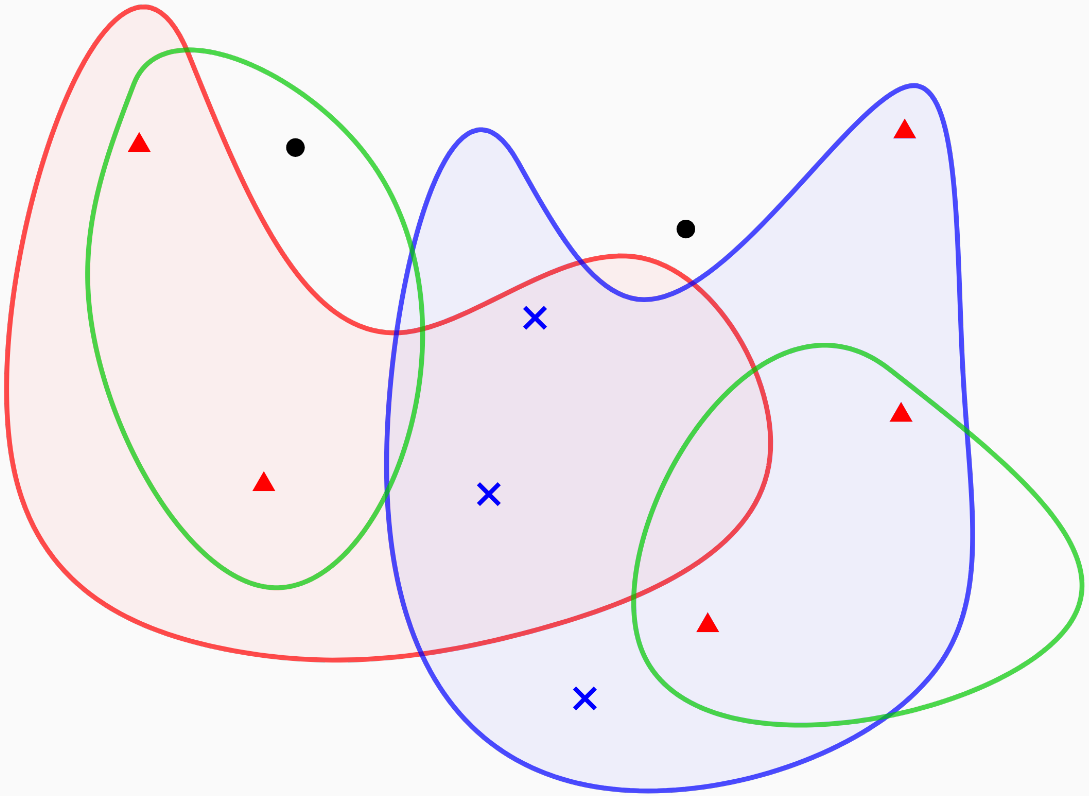

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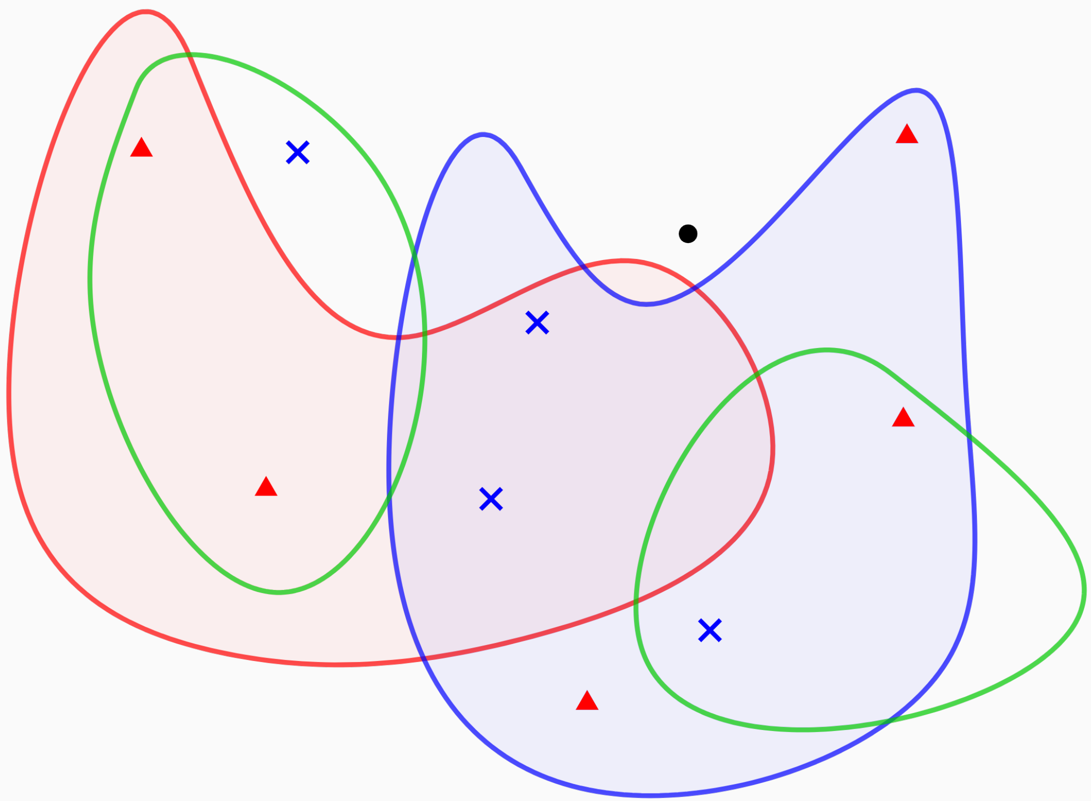

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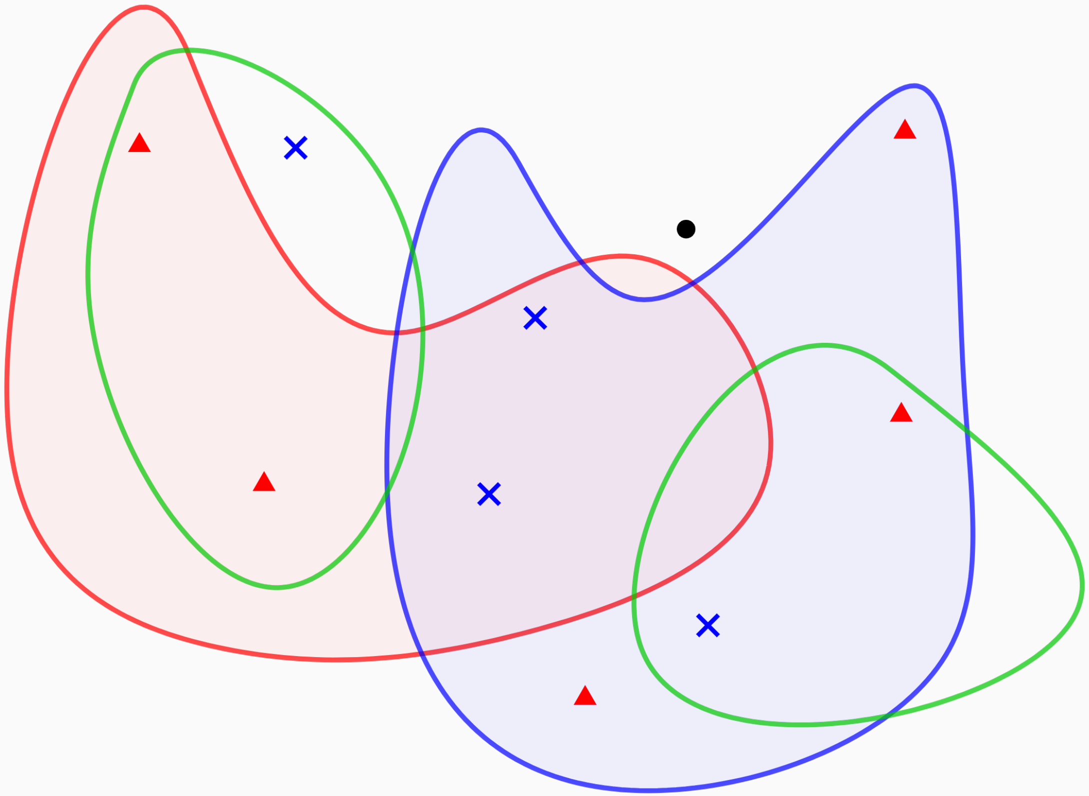

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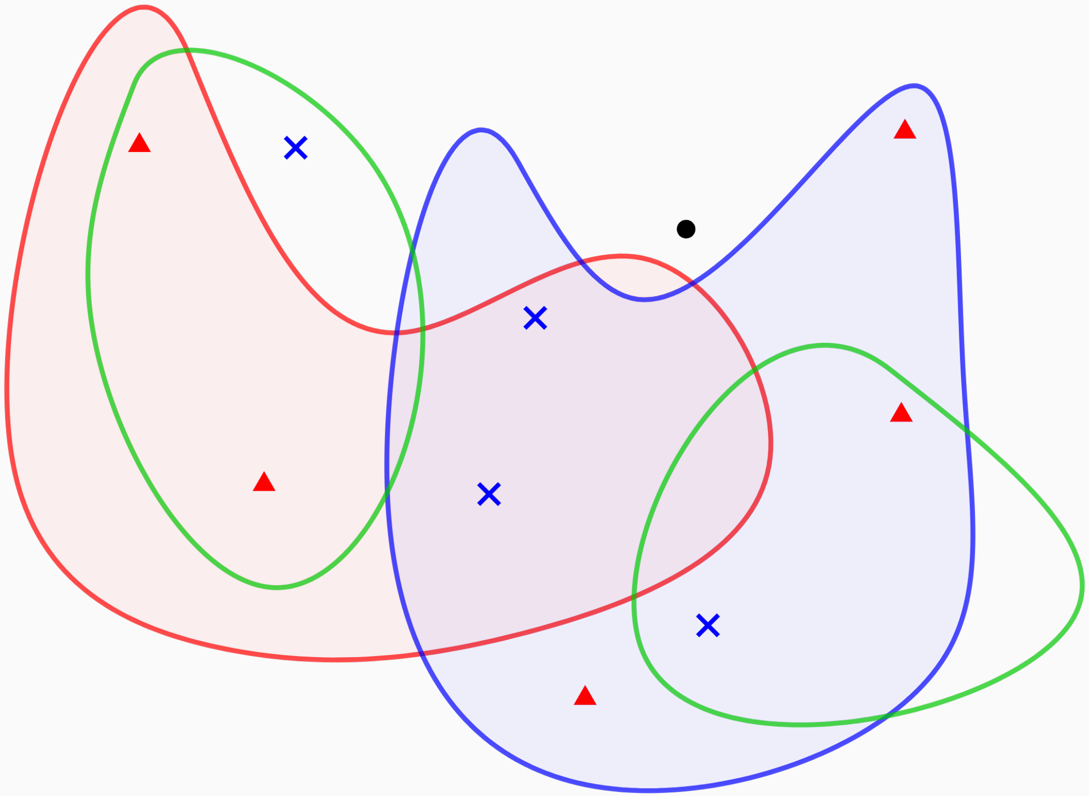

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Hard for $m = \Omega(n)$: $\Omega\left(\sqrt{n \log\left(\frac{m}{n}\right)}\right)$ in general, $\Omega\left(n^{\frac{1}{2}-\frac{1}{2d}}\right)$ for VC-dim d

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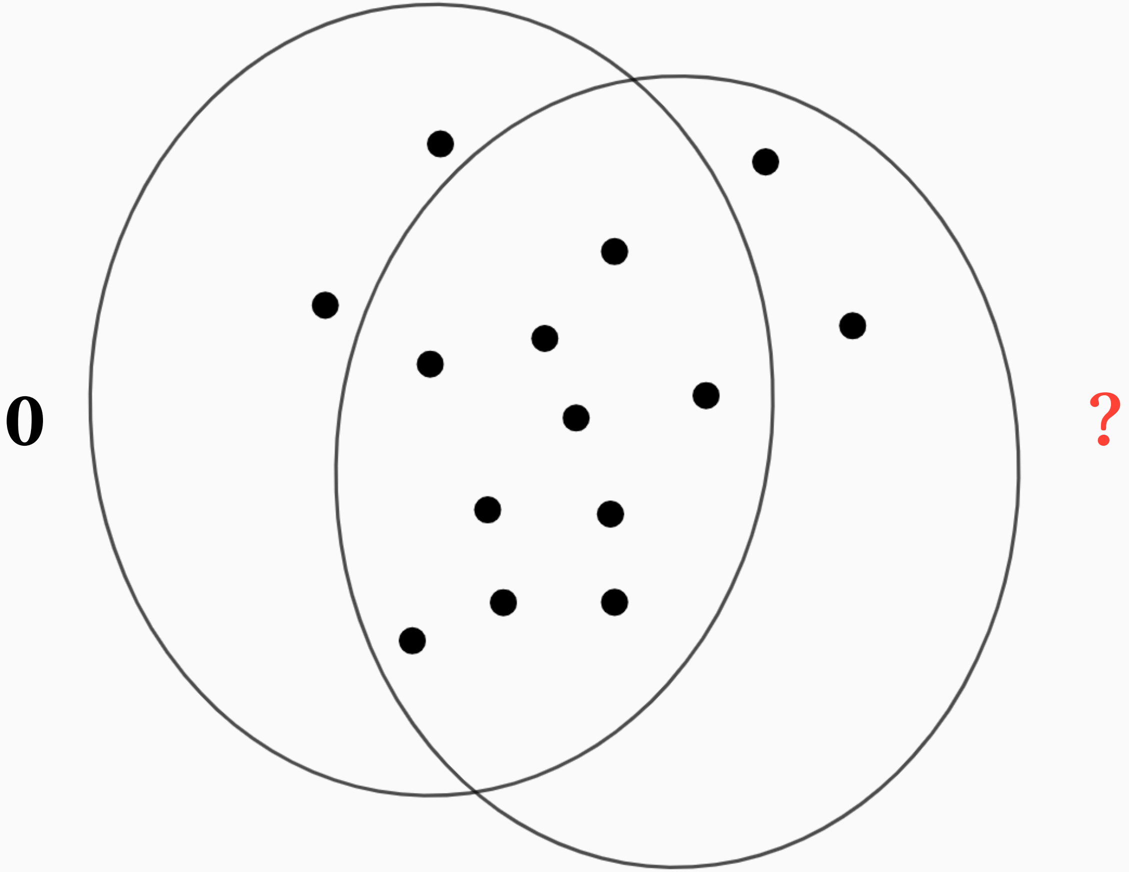

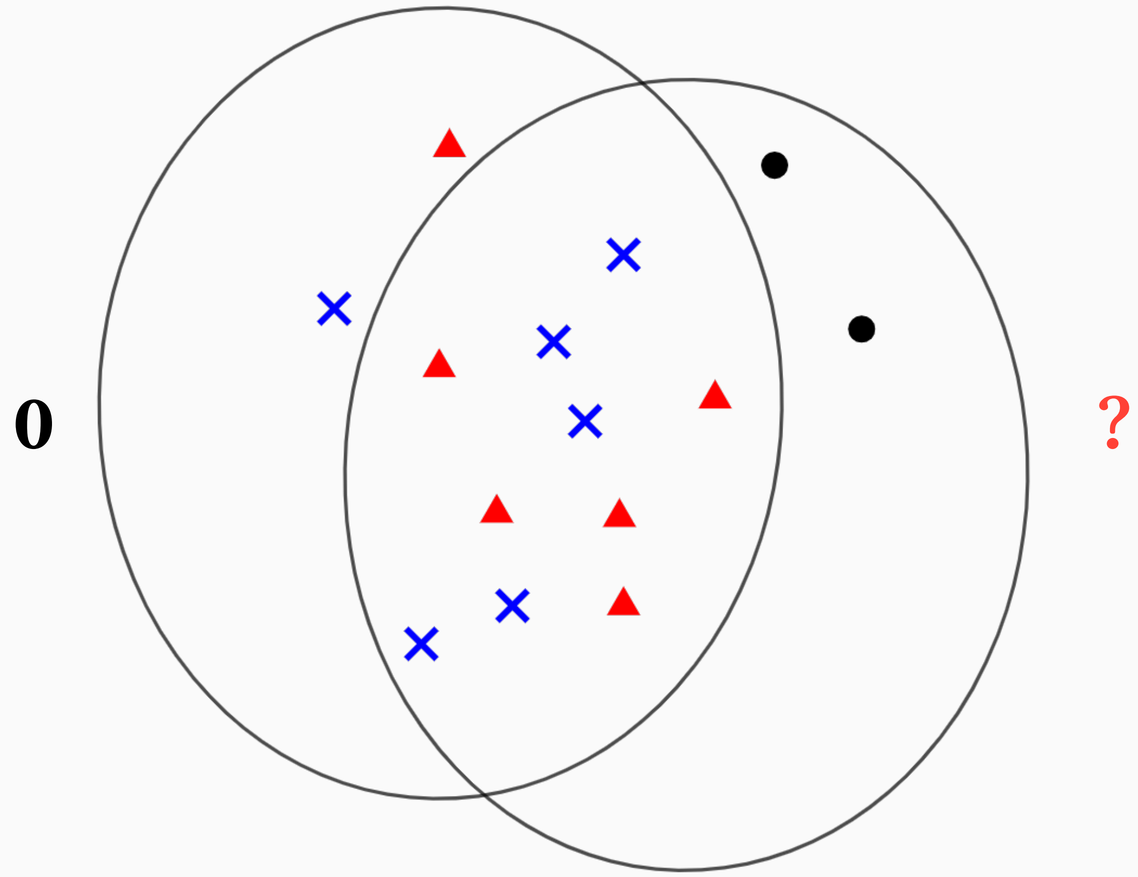
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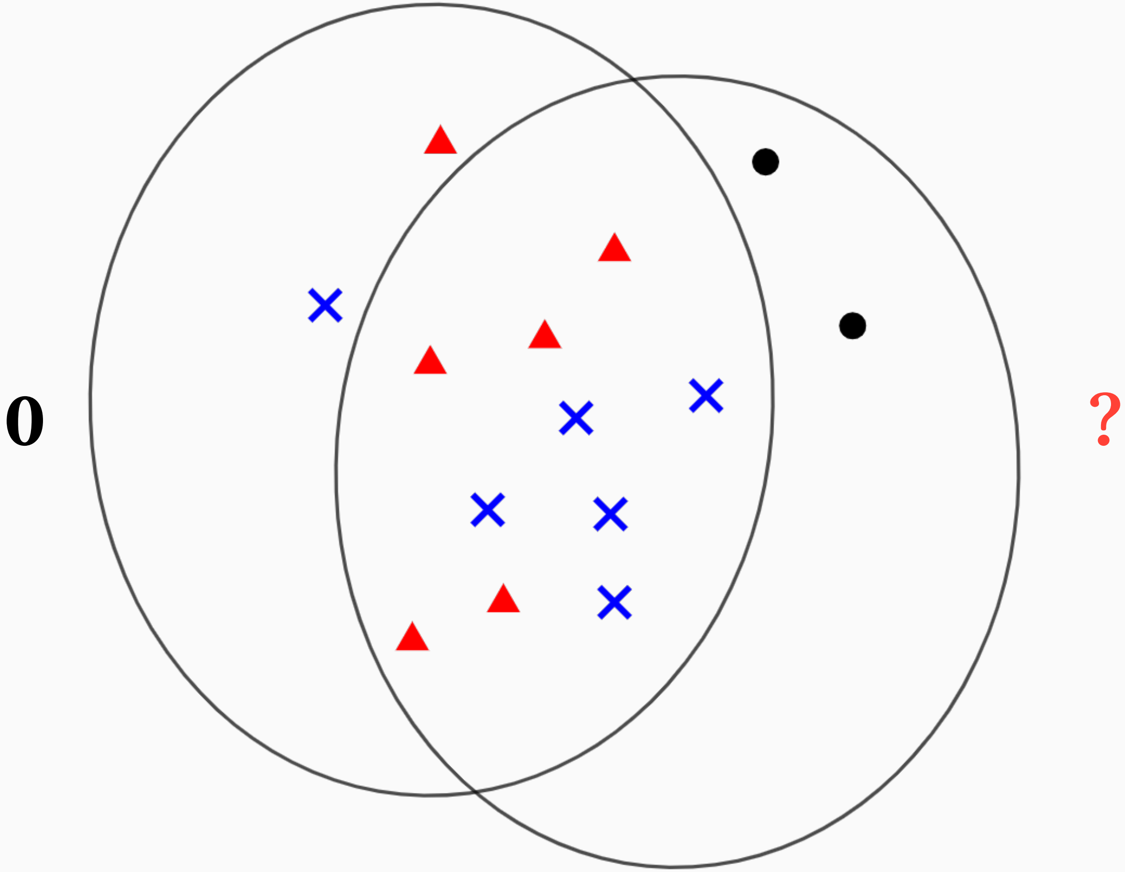

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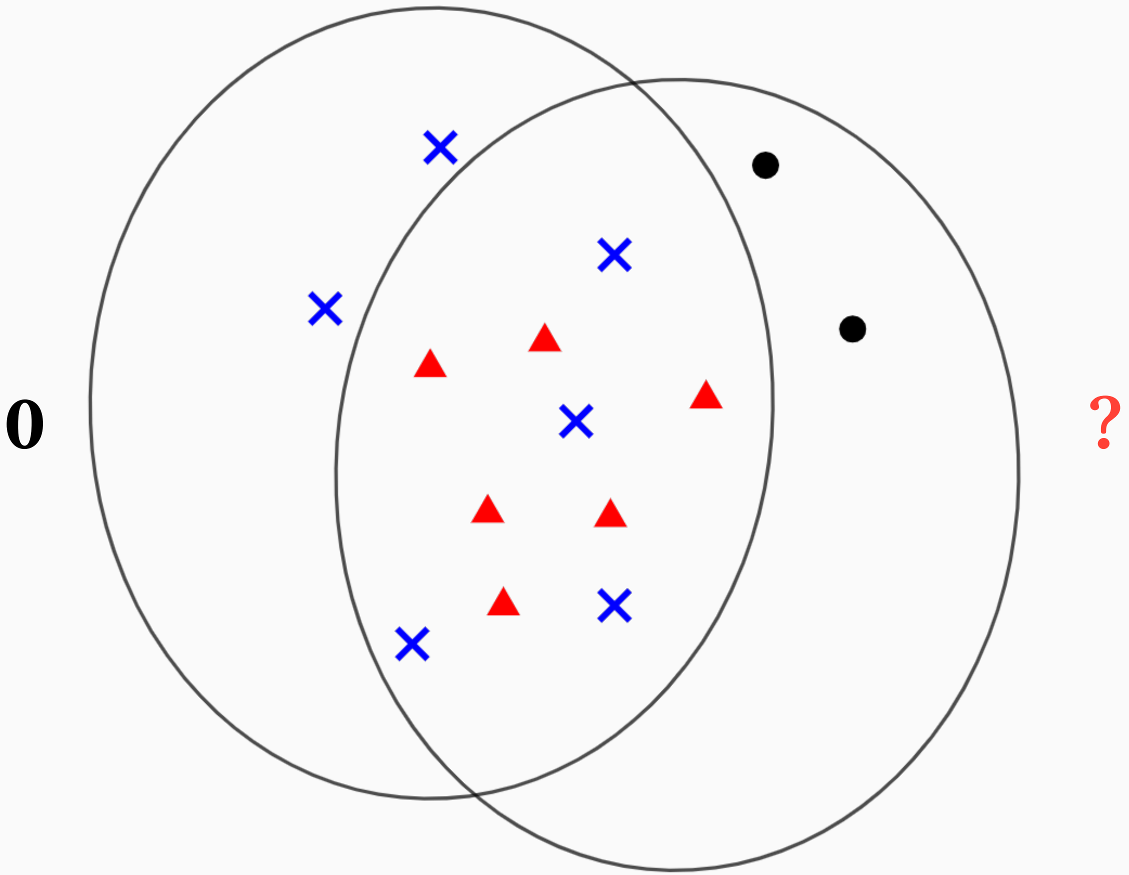

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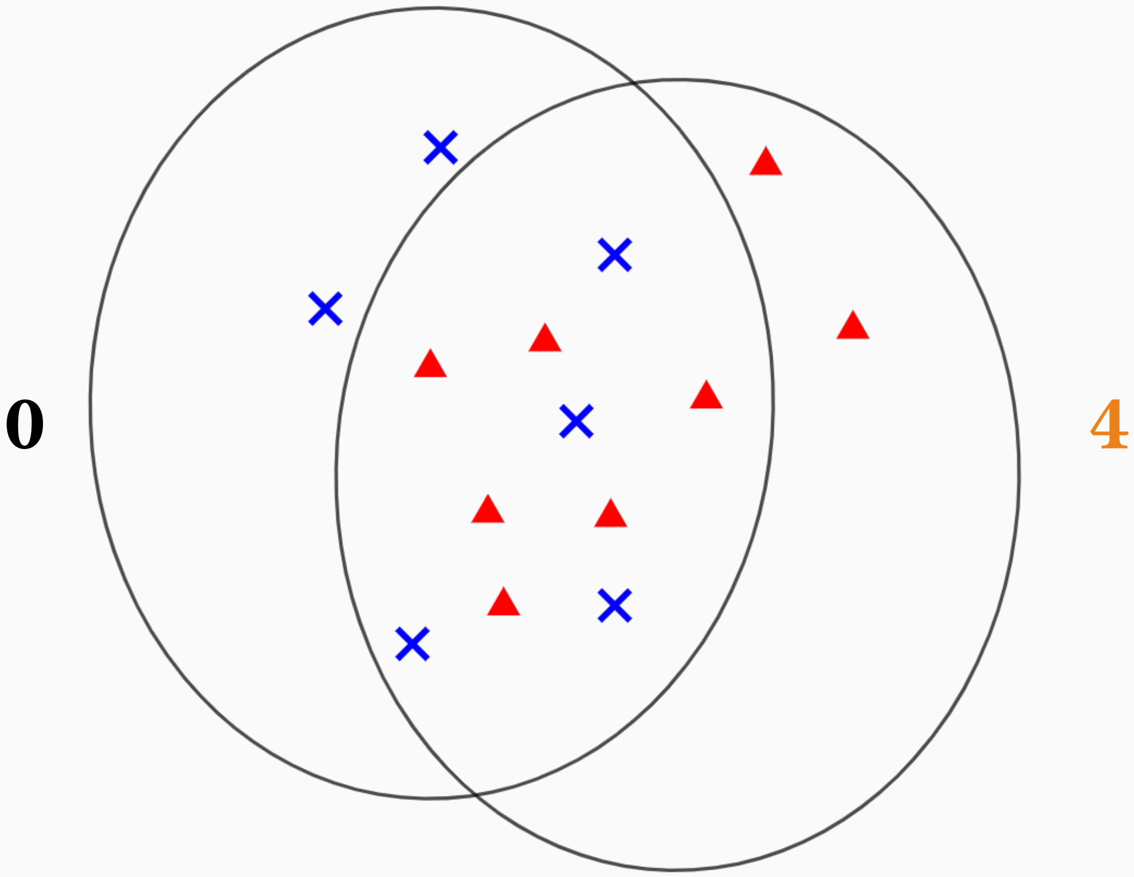

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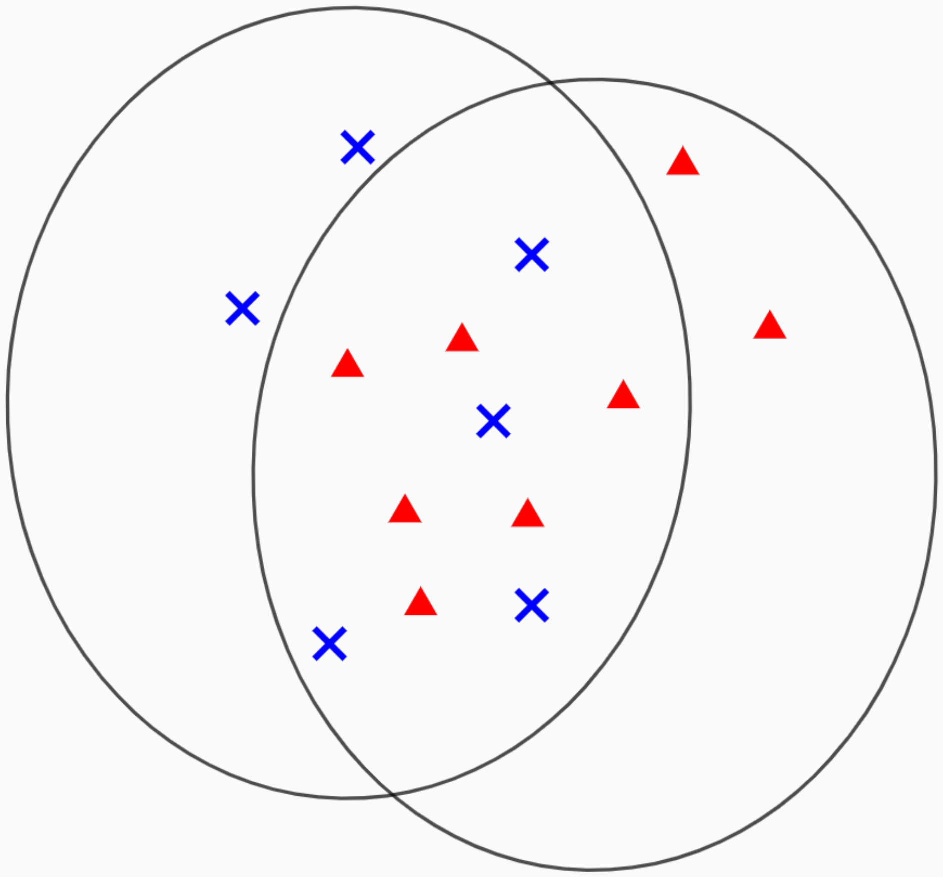

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Can be polynomially computed starting from a random sample of X (ε-net) (Matousek et al. 1993, Louvet 2025)

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With a little more effort (chaining), one can obtain $O\left(n^{\frac{1}{2}-\frac{1}{2d}}\right)$ (optimal)

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Differential privacy?

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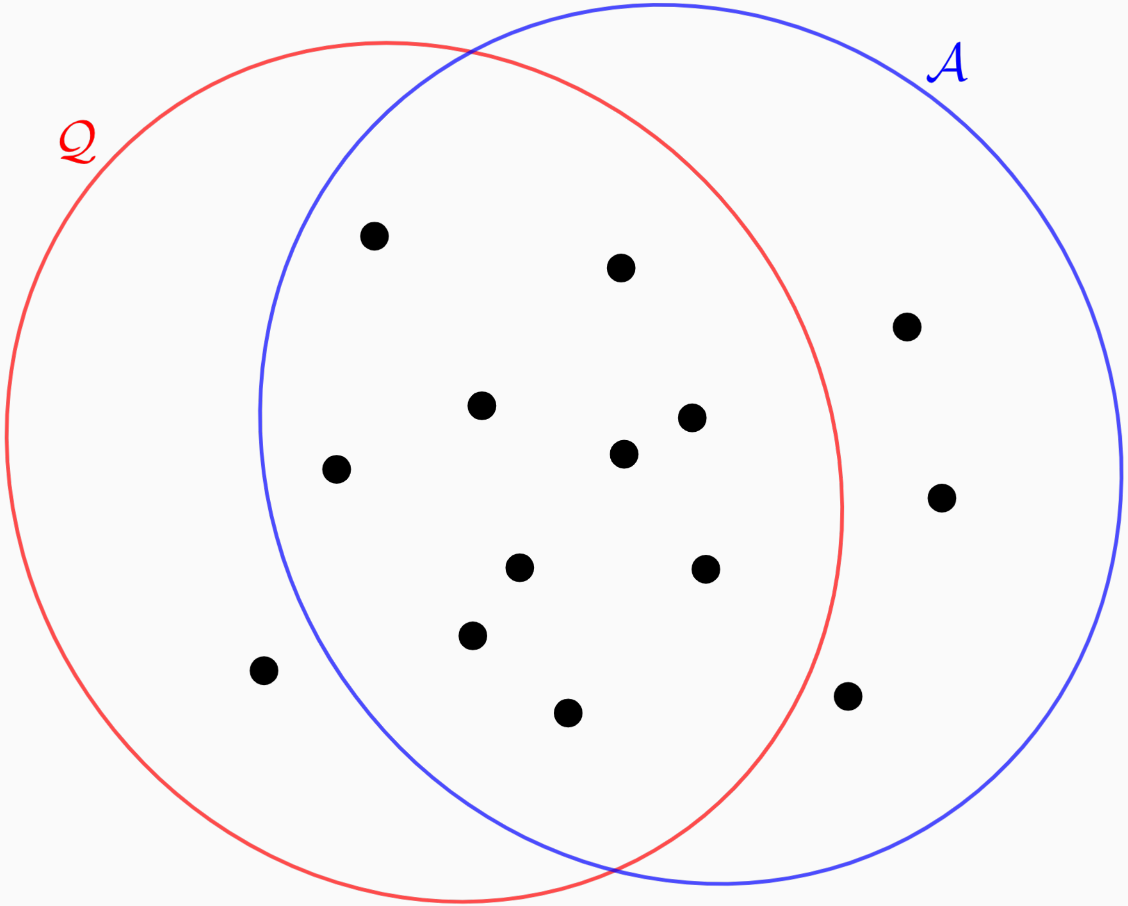

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Prospective DP paradigm:

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Thank you.

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